Introduction

This is a very classical parabolic Robin boundary control problem in one space dimension with control constraints. Originally it was posed as a time-optimal control problem, see Schittkowski [1979]. Several years later in Tröltzsch [1984] this problem was modified to an optimal control problem with fixed final time. In Eppler and Tröltzsch [1986] an additional Tikhonov regularization was introduced. We present the example in that form but change the notation to standard variables. The same example is studied in many publications sometimes with small modifications. If the regularization parameter α is zero, then the optimal control has bang-bang structure with one switching point.

Variables & Notation

Unknowns

$u \in L^2(0,T)$	control variable
$y \in C([0,T], L^2(0,1))$	state variable

Given Data

No analytic solution is known for the given data. The numerical experiments in the literature show results for $\alpha = 0$ (no regularization) with a bang-bang structure as well as results for $\alpha > 0$. In particular one can find results for $\alpha = 10^{-\nu}$ with different ν in Eppler and Tröltzsch [1986].

$\Omega = (0,1)$	computational domain
$T = 1.58$	final time
$y_d(x) = \frac{1}{2}(1 - x^2)$	desired final state
$b = 1$	heat exchange coefficient
$\alpha \geq 0$	Tikhonov parameter

Problem Description

$$\text{Minimize} \quad \frac{1}{2}\|y(T,\cdot) - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(0,T)}$$

$$\text{s.t.} \quad \begin{cases} \dfrac{\partial y}{\partial t} - \triangle y = 0 & \text{in } (0,T)\times\Omega \\ y(0,x) = 0 & \text{in } \Omega \\ \dfrac{\partial y}{\partial x}(t,0) = 0 & \text{in } (0,T) \\ \dfrac{\partial y}{\partial x}(t,1) = b(u(t) - y(t,1)) & \text{in } (0,T) \end{cases}$$

$$\text{and} \quad -1 \le u(t) \le 1 \quad \text{in } (0,T).$$

Optimality System

The following optimality system for the state $y \in C([0,T], L^2(0,1))$, the control $u \in L^2(0,T)$ and the adjoint state $p \in C([0,T], L^2(0,1))$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned} &\frac{\partial y}{\partial t} - \triangle y = 0 && \text{in } (0,T)\times\Omega && -\frac{\partial p}{\partial t} - \triangle y = 0 && \text{in } (0,T)\times\Omega \\ &y(0,x) = 0 && \text{in } \Omega && p(T,x) = y(T,x) - y_d(x) && \text{in } \Omega \\ &\frac{\partial y}{\partial x}(t,0) = 0 && \text{in } (0,T) && \frac{\partial p}{\partial x}(t,0) = 0 && \text{in } (0,T) \\ &\frac{\partial y}{\partial x}(t,1) = b(u(t) - y(t,1)) && \text{in } (0,T) && \frac{\partial p}{\partial x}(t,1) = -b\cdot p(t,1) && \text{in } (0,T) \end{aligned}$$

as well as

$$\int_0^T \big(p(t,1) + \alpha\, u(t)\big)\big(v(t) - u(t)\big)\,\mathrm{d}t \ge 0 \quad \text{for all } v \in L^2(0,T) \text{ with } |v(t)| \le 1 \text{ a.e.}$$

In case $\alpha > 0$, this variational inequality is equivalent to the projection formula

$$u(t) = \operatorname{proj}_{[-1,1]}\left(-\frac{p(t,1)}{\alpha}\right).$$

Supplementary Material

There is no analytic solution known. An interesting modification is to take a desired state

$$y_d(x) = \tfrac{1}{2}(1-x).$$

Then the optimal control oscillates when approaching the final time T.

References

K. Eppler and F. Tröltzsch. On switching points of optimal controls for coercive parabolic boundary control problems. *Optimization. A Journal of Mathematical Programming and Operations Research*, 17(1):93–101, 1986. ISSN 0233-1934. doi: 10.1080/02331938608843105.

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