Introduction

This is a boundary optimal control problem governed by a quasilinear elliptic equation. Problems involving quasilinear equations are particularly challenging with regard to their analysis, numerical analysis, and numerical solution.

This problem and the analytical example were published as the first example in Casas and Dhamo [2012] and in [Dhamo, 2012, Example 4.14].

Variables & Notation

Unknowns

$$
\begin{aligned}
u &\in L^2(\Gamma) && \text{control variable} \\
y &\in H^{3/2}(\Omega) && \text{state variable}
\end{aligned}
$$

Given Data

The given data is chosen in a way which admits an analytic solution.

$$
\begin{aligned}
\Omega &= (0,\pi)^2 && \text{computational domain} \\
&\Gamma && \text{its boundary} \\
x &= (x_1, x_2) && \text{coordinate in } \Omega \\
\alpha &= -20 && \text{lower control bound} \\
\beta &= -2 && \text{upper control bound} \\
a(x,y) &= 1 + (x_1+x_2)^2 + y^2 && \text{leading coefficient} \\
f(x,y) &= 2\,y\,\big(\sin^2(x_1) + \sin^2(x_2)\big) && \text{potential term} \\
g_1(x) &= 2\,\sin(x_1)\,\sin(x_2)\big(1+(x_1+x_2)^2\big) + 6\,\sin^3(x_1)\,\sin^3(x_2) \\
&\quad - (x_1+x_2)\big(\sin(x_1)\,\cos(x_2) + \cos(x_1)\,\sin(x_2)\big) && \text{right hand side} \\
g_2(x) &= \min\{0, e(x)-\alpha\} + \max\{0, e(x)-\beta\} && \text{boundary right hand side} \\
e(x) &= -\big(1+(x_1+x_2)^2\big)\,\sin(x_1+x_2)^2 && \text{auxiliary term} \\
\eta(x) &= -1 - \operatorname{proj}_{[\alpha,\beta]} e(x) && \text{coefficient in the objective} \\
y_\Omega(x) &= \sin(x_1)\,\sin(x_2) - 2\,\big(\sin^2(x_1) + \sin^2(x_2)\big) && \text{desired state}
\end{aligned}
$$

Note that the formula for the auxiliary term $e(x)$ above has been corrected (a square was missing). The formula originally given in [Casas and Dhamo, 2012, p.753] and [Dhamo, 2012, p.117] does not match the function shown in [Dhamo, 2012, Figure 4.1]. Preference has been given to the reproduction of the figure.

Problem Description

$$
\begin{aligned}
&\text{Minimize} \quad \frac{1}{2}\|y-y_\Omega\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Gamma)} + \int_\Gamma \eta\, u \,\mathrm{d}s \\
&\text{s.t.} \quad \begin{cases} -\operatorname{div}\big[a(x,y)\,\nabla y\big] + f(x,y) = g_1 & \text{in } \Omega \\ a(x,y)\,\dfrac{\partial y}{\partial n} = u + g_2 & \text{on } \Gamma \end{cases} \\
&\text{and} \quad \alpha \le u \le \beta \quad \text{on } \Gamma
\end{aligned}
$$

Optimality System

The following optimality system for the state $y \in H^{3/2}(\Omega)$, the control $u \in L^2(\Gamma)$ and the adjoint state $\varphi \in H^{3/2}(\Omega)$, given in the strong form, represents a set of first-order necessary optimality conditions.

$$
\begin{aligned}
-\operatorname{div}\big[a(x,y)\,\nabla y\big] + f(x,y) &= g_1 && \text{in } \Omega \\
a(x,y)\,\frac{\partial y}{\partial n} &= u + g_2 && \text{on } \Gamma \\
-\operatorname{div}\big[a(x,y)\,\nabla \varphi\big] + \frac{\partial a}{\partial y}(x,y)\nabla y \cdot \nabla\varphi + \frac{\partial f}{\partial y}(x,y)\,\varphi &= y - y_\Omega && \text{in } \Omega \\
a(x,y)\,\frac{\partial \varphi}{\partial n} &= 0 && \text{on } \Gamma \\
u &= \operatorname{proj}_{[\alpha,\beta]}(-\eta - \varphi|_\Gamma) && \text{on } \Gamma
\end{aligned}
$$

Supplementary Material

The following state, control and adjoint state variables are shown in Casas and Dhamo [2012] to satisfy first-order necessary conditions. Moreover, second-order sufficient conditions also hold due to the structure of the objective. Consequently, u is a local minimum (in the sense of $L^\infty(\Gamma)$).

$$
\begin{aligned}
y &= \sin(x_1)\,\sin(x_2) && \text{in } \Omega \\
\varphi &= 1 && \text{in } \Omega \\
u &= \operatorname{proj}_{[\alpha,\beta]} e = -1 - \eta && \text{on } \Gamma
\end{aligned}
$$

This solution is particular in the sense that the upper and lower bound constraints for the control are both active on nontrivial parts of the boundary, but never strongly active. In other words, strict comlementarity does not hold for this problem. A plot of the optimal control is provided in [Dhamo, 2012, Figure 4.1].

References

E. Casas and V. Dhamo. Error estimates for the numerical approximation of Neumann control problems governed by a class of quasilinear elliptic equations. *Computational Optimization and Applications. An International Journal*, 52(3):719–756, 2012. ISSN 0926-6003. doi: 10.1007/s10589-011-9440-0.

V. Dhamo. *Optimal Boundary Control of Quasilinear Partial Differential Equations: Theory and Numerical Analysis*. PhD thesis, Technische Universität Berlin, 2012. URL http://opus.kobv.de/tuberlin/volltexte/2012/3511.