Introduction

We study the optimal control of a particular gradient enhanced damage model. The damage model is based on Dimitrijevic and Hackl [2008], the optimization problem is currently unpublished and provided by M. Holtmannspötter.

The presented damage model features two damage variables, one with higher spatial regularity and one which carries the evolution of damage in time. Their difference is penalized in the free energy functional. The evolution of damage in time is modeled by a nonsmooth operator ODE. Therefore the control-to-state operator is not differentiable whenever there are *biactive* points present.

This example features three sets of data, such that the known global optimum has either only inactive points, only strongly active points, or only biactive points.

Variables & Notation

Unknowns

In contrast to the original damage model in Dimitrijevic and Hackl [2008], the situation is simplified by considering only scalar valued functions. The unknown functions are

$$
\begin{aligned}
\ell &\in L^2((0,T),L^2(\Omega)) && \text{control variable (body force)}\\
u &\in L^2((0,T),H_0^1(\Omega)) && \text{state variable (displacement)}\\
\varphi &\in L^2((0,T),H^1(\Omega)) && \text{state variable (nonlocal damage)}\\
d &\in H^1((0,T),L^2(\Omega)) && \text{state variable (local damage).}
\end{aligned}
$$

Given Data

For the construction of analytically known solutions, the following functions need to be specified:

$$
\begin{aligned}
\ell_d &\in L^2((0,T),L^2(\Omega)) && \text{desired control}\\
u_d &\in L^2((0,T),L^2(\Omega)) && \text{desired displacement}\\
\varphi_d &\in L^2((0,T),L^2(\Omega)) && \text{desired nonlocal damage}\\
d_d &\in L^2((0,T),L^2(\Omega)) && \text{desired local damage}\\
e_1 &\in L^2((0,T),L^2(\Omega)) && \text{auxiliary body force}\\
e_2 &\in L^2((0,T),H^1(\Omega)) && \text{auxiliary forcing in the nonlocal damage}\\
d_0 &\in L^2(\Omega) && \text{initial local damage.}
\end{aligned}
$$

These data will be specified below in the supplementary materials section, such that globally optimal solutions with the desired properties are known. In addition, the following data are needed to specify all variables in the problem:

$T = 1$	final time
$\Omega = (0,1)^2$	computational domain
$\alpha = 1$	nonlocal damage parameter
$\beta = 10^4$	penalty parameter for coupling of d and φ
$\delta = 1$	viscosity parameter in the damage evolution
$r = 1$	fracture toughness
$\eta = 10^{-2}$	regularization parameter to avoid material degeneration
$g(z) = (1-\eta)\, e^{-z} + \eta$	C^2 function satisfying $g(0) = 1$ and $\lim\limits_{z\to\infty} g(z) = \eta$
$g'(z) = \eta\, e^{-z}$	derivative of the above function
$g''(z) = -\eta\, e^{-z}$	derivative of the above function

Additional Notation

In this example, the difficulty lies in the potential non-differentiability of the $\max\{0,\cdot\}$ operator in the evolution law of the local damage variable. To describe the area where differentiability is critical, we define three sets:

$$\begin{aligned}
A(t) &= \{x \in \Omega : -\beta(d(x,t) - \varphi(x,t)) - r > 0\} && \text{the strongly active set}\\
B(t) &= \{x \in \Omega : -\beta(d(x,t) - \varphi(x,t)) - r = 0\} && \text{the biactive set}\\
I(t) &= \{x \in \Omega : -\beta(d(x,t) - \varphi(x,t)) - r < 0\} && \text{the inactive set}
\end{aligned}$$

Problem Description

We use a standard tracking type functional:

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2}\|u - u_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\|\varphi - \varphi_d\|^2_{L^2(0,T;L^2(\Omega))}\\
& + \frac{1}{2}\|d - d_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{1}{2}\|\ell - \ell_d\|^2_{L^2(0,T;L^2(\Omega))}.
\end{aligned}$$

The above minimization is subject to the constraints

$$
\begin{aligned}
\int_\Omega g(\varphi(t))\,\nabla u(t)\cdot\nabla v\,\mathrm{d}x &= \int_\Omega (\ell(t)+e_1(t))\,v\,\mathrm{d}x && \forall v\in H_0^1(\Omega)\\
\int_\Omega \alpha\nabla\varphi(t)\cdot\nabla\psi+\beta\,\varphi(t)\,\psi+\frac{1}{2}g'(\varphi(t))\nabla u(t)\cdot\nabla u(t)\,\psi\,\mathrm{d}x & \\
&= \int_\Omega \beta\,d(t)\,\psi\,\mathrm{d}x+\int_\Omega e_2(t)\,\psi\,\mathrm{d}x && \forall\psi\in H^1(\Omega)\\
\dot d(t) &= \frac{1}{\delta}\max\{-\beta(d(t)-\varphi(t))-r,0\} && \text{a.e. in }\Omega
\end{aligned}
$$

for almost all $t\in(0,T)$, as well as

$$
d(0)=d_0 \qquad \text{a.e. in }\Omega.
$$

The functions e_1, e_2, and d_0 in the system are inserted to allow the construction of a known solution for the optimization problem. The function e_1 can be interpreted as a given (uncontrolled) load and d_0 as initial local damage.

Optimality System

The control-to-state operator is, in general, not differentiable. Consequently, standard methods for the derivation of necessary optimality conditions using adjoint techniques fail. If, however, the biactive set $B(t)$ is a set of measure zero a.e. in $(0,T)$, then the directional derivative of the control-to-state map is linear, and adjoint states can be introduced. In this case, first order necessary optimality conditions for the above problem in a point (u,φ,d,ℓ) are given by the existence of p_1, p_2, p_3 such that the following adjoint system

$$
\begin{aligned}
\int_\Omega g(\varphi(t))\nabla p_1(t)\cdot\nabla v\,\mathrm{d}x+\int_\Omega g'(\varphi(t))\,p_2(t)\nabla u(t)\cdot\nabla v\,\mathrm{d}x & \\
= \int_\Omega (u(t)-u_d(t))\,v\,\mathrm{d}x & && \forall v\in H_0^1(\Omega)\\
\int_\Omega \alpha\nabla p_2(t)\cdot\nabla\psi+\beta\left(p_2(t)-\frac{1}{\delta}\chi_{A(t)}\,p_3(t)\right)\psi+g'(\varphi(t))\nabla u(t)\cdot\nabla p_1(t)\,\psi\,\mathrm{d}x & \\
+\int_\Omega \frac{1}{2}g''(\varphi(t))\,p_2(t)\,\nabla u(t)\cdot\nabla u(t)\,\psi\,\mathrm{d}x & \\
= \int_\Omega (\varphi(t)-\varphi_d(t))\,\psi\,\mathrm{d}x & && \forall\psi\in H^1(\Omega)\\
-\dot p_3(t) = \beta\left(p_2(t)-\frac{1}{\delta}\chi_{A(t)}\,p_3(t)\right)+d(t)-d_d(t) & && \text{a.e. in }\Omega
\end{aligned}
$$

for almost all $t \in (0, T)$, as well as

$$p_3(T) = 0 \qquad \text{a.e. in } \Omega.$$

Moreover, the gradient equation

$$p_1 + \ell - \ell_d = 0$$

holds a.e. in $(0, T) \times \Omega$.

Supplementary Material

In this section, we provide three different sets of data, leading to the three distinct cases featuring only inactive points, only active points, and only biactive points. For all three settings, we define the auxiliary functions

$$\begin{aligned}
e_1(x,t) &= -g(\varphi_d(x,t))\,\triangle u_d(x,t) - g'(\varphi_d(x,t))\,\nabla u_d(x,t)\cdot\nabla\varphi_d(x,t) - \ell_d(x,t),\\
e_2(x,t) &= -\alpha\,\triangle\varphi_d(x,t) + \beta\,(\varphi_d(x,t) - d_d(x,t)) + \frac{1}{2}g'(\varphi_d(x,t))\,|\nabla u_d(x,t)|^2,\\
d_0(x) &= d_d(x,0).
\end{aligned}$$

In virtue of this construction, the unique *global* optimum is $u = u_d$, $\varphi = \varphi_d$, $d = d_d$, and $\ell = \ell_d$ and consequently the adjoint state is $(p_1, p_2, p_3) \equiv (0, 0, 0)$ in all three cases. Clearly, the corresponding value of the objective is zero.

Case 1: Only inactive points ($I(t) = \Omega$)

$$\begin{aligned}
\ell_d(x,t) &= \sin(\pi x_1)\sin(\pi x_2)\\
u_d(x,t) &= \frac{1}{2\pi^2}\sin(\pi x_1)\sin(\pi x_2)\\
\varphi_d(x,t) &= 0\\
d(x,t) &= 0
\end{aligned}$$

Case 2: Only strongly active points ($A(t) = \Omega$)

$$\begin{aligned}
\ell_d(x,t) &= \sin(\pi x_1)\sin(\pi x_2)\\
u_d(x,t) &= \frac{1}{2\pi^2}\sin(\pi x_1)\sin(\pi x_2)\\
\varphi_d(x,t) &= \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{3}(x_1^3 + x_2^3) + 1\\
d_d(x,t) &= \left(\varphi_d(x,t) - \frac{r}{\beta}\right)\left(1 - e^{-\frac{\beta}{\delta}t}\right)
\end{aligned}$$

Notice that in this case, the function $e^{-\frac{\beta}{\delta}t} = e^{-10^4 t} \approx 0$.

Case 3: Only biactive points ($B(t) = \Omega$)

$$
\begin{aligned}
\ell_d(x,t) &= \sin(\pi x_1)\sin(\pi x_2) \\
u_d(x,t) &= \frac{1}{2\pi^2}\sin(\pi x_1)\sin(\pi x_2) \\
\varphi_d(x,t) &= \frac{1}{2\pi^2}\cos(\pi x_1)\cos(\pi x_2) \\
d_d(x,t) &= \varphi_d(x,t) - \frac{r}{\beta}
\end{aligned}
$$

References

B. J. Dimitrijevic and K. Hackl. A method for gradient enhancement of continuum damage models. *Technische Mechanik*, 28(1):43–52, 2008. URL http://www.uni-magdeburg.de/ifme/zeitschrift_tm/2008_Heft1/05_Dimitrievich_Hackl.pdf.