Introduction

This is one of the simplest model problems in optimal control of partial differential equations. Problems of this type are treated extensively in [Tröltzsch, 2010, Chapter 2], and are sometimes refered to as the *mother problem* type. The present problem is special in the sense that the control acts in a distributed way on the entire domain Ω, and that the state is observed on the entire domain as well. Furthermore, no constraints beside the elliptic PDE are present.

This problem was adapted from [Tröltzsch, 2010, Section 2.9.1], where the case $\nu = 0$ with additional control constraints was elaborated.

Variables & Notation

Unknowns

$u \in L^2(\Omega)$	control variable
$y \in H_0^1(\Omega)$	state variable

Given Data

The given data is chosen in a way which admits an analytic solution.

$\Omega = (0,1)^2$	computational domain
$\nu = 10^{-2}$	control cost parameter
$y_d = -\sin(8\,\pi\,x_1)\,\sin(8\,\pi\,x_2) + \sin(\pi\,x_1)\,\sin(\pi\,x_2)$	desired state
$f = 2\,\pi^2\sin(\pi\,x_1)\,\sin(\pi\,x_2) + \dfrac{\nu^{-1}}{128\,\pi^2}\sin(8\,\pi\,x_1)\,\sin(8\,\pi\,x_2)$	uncontrolled force

Problem Description

$$\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}$$

$$\text{s.t.} \quad \begin{cases} -\triangle y = u + f & \text{in } \Omega \\ \quad y = 0 & \text{on } \partial\Omega \end{cases}$$

Optimality System

The following optimality system for the state $y \in H_0^1(\Omega)$, the control $u \in L^2(\Omega)$ and the adjoint state $p \in H_0^1(\Omega)$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned} -\triangle y &= u + f && \text{in } \Omega \\ y &= 0 && \text{on } \partial\Omega \\ -\triangle p &= -(y - y_d) && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega \\ \nu\, u - p &= 0 && \text{in } \Omega \end{aligned}$$

Supplementary Material

The optimal state, adjoint state and control are known analytically:

$$\begin{aligned} y &= \sin(\pi\, x_1)\, \sin(\pi\, x_2) \\ p &= -\frac{1}{128\,\pi^2} \sin(8\,\pi\, x_1)\, \sin(8\,\pi\, x_2) \\ u &= -\frac{\nu^{-1}}{128\,\pi^2} \sin(8\,\pi\, x_1)\, \sin(8\,\pi\, x_2) \end{aligned}$$

References

F. Tröltzsch. *Optimal Control of Partial Differential Equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, 2010.