Introduction

Here we present a distributed optimal control problem of the time-dependent Stokes equations. The problem was derived as a test for the paper Güttel and Pearson [2017], which required optimal states and controls that are not polynomial in spatial or time variables. The problem maintains a parameter dependence for the regularization parameter β to serve as a test case for the β dependence of solvers. This problem and its analytical solution appear in [Güttel and Pearson, 2017, Section 6.2], with computations for final time $T = 1$ and control cost parameter $\beta = 10^{-2}$.

Variables & Notation

Unknowns

$\boldsymbol{u} \in L^2(\Omega \times I)^2$	control variable
$\boldsymbol{y} = (\boldsymbol{v}, p)$	state variable (velocity, pressure)
$\boldsymbol{v} \in L^2(I, V(\Omega)) \cap H^1(I, V(\Omega)^*)$	velocity component of state
$p \in L^2(I, L^2_0(\Omega))$	pressure component of state

where

$$V(\Omega) = \left\{\boldsymbol{v} \in H^1_0(\Omega)^2 : \operatorname{div} \boldsymbol{v} = 0\right\},$$
$$L^2_0(\Omega) = \left\{p \in L^2(\Omega) : \int_\Omega p \, \mathrm{d}x\right\}.$$

Given Data

$T > 0$	length of time interval
$\Omega = (-1,1)^2$	spatial domain
$I = (0,T)$	time interval
$Q = (-1,1)^2 \times (0,T)$	space-time domain
$\Sigma = \partial\Omega \times (0,T)$	lateral boundary of Q
$\beta > 0$	regularization parameter
$\zeta = -\dfrac{e^T}{4\,\pi^2\,\beta}$	parameter
$\eta = \dfrac{1}{(1+4\,\pi^2)\,\beta}$	parameter
$\boldsymbol{z} = \left[-4\,\pi^2\,(\zeta+\eta\,e^t)\,\cos(2\,\pi\,x_1)\sin(2\,\pi\,x_2),\;0\right]^\top$	uncontrolled source term
$\boldsymbol{v}_d = [v_{d,1},\;v_{d,2}]^\top$	desired state
$v_{d,1} = 1 + \left(-e^t + (\zeta+\eta\,e^t)\right)\,\sin^2(\pi\,x_1)\sin(2\,\pi\,x_2) + 2\,\pi^2\,(e^T - e^t)\,\left(1 - 4\,\sin^2(\pi\,x_1)\right)\,\sin(2\,\pi\,x_2)$	component of desired state
$v_{d,2} = 1 + \left(e^t - (\zeta+\eta\,e^t)\right)\,\sin(2\,\pi\,x_1)\,\sin^2(\pi\,x_2) + 2\,\pi^2\,(e^T - e^t)\,\sin(2\,\pi\,x_1)\,\left(4\,\sin^2(\pi\,x_2) - 1\right)$	component of desired state
$\boldsymbol{v}_0 = [v_{0,1},\;v_{0,2}]^\top$	initial state
$v_{0,1} = (\zeta+\eta)\,\sin^2(\pi\,x_1)\,\sin(2\,\pi\,x_2)$	component of initial state
$v_{0,2} = -(\zeta+\eta)\,\sin(2\,\pi\,x_1)\,\sin^2(\pi\,x_2)$	component of initial state

Problem Description

$$\text{Minimize}\quad \frac{1}{2}\int_I\int_\Omega |\boldsymbol{v}-\boldsymbol{v}_d|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2}\int_I\int_\Omega |\boldsymbol{u}|^2\,\mathrm{d}x\,\mathrm{d}t$$

$$\text{s.t.}\quad \begin{cases} \boldsymbol{v}_t - \triangle\boldsymbol{v} + \nabla p = \boldsymbol{u} + \boldsymbol{z} & \text{in } Q \\ -\operatorname{div}\boldsymbol{v} = 0 & \text{in } Q \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Sigma \\ \boldsymbol{v} = \boldsymbol{v}_0 & \text{at } t = 0 \end{cases}$$

Optimality System

The following optimality system for the state $\boldsymbol{y} = (\boldsymbol{v}, p) \in L^2(I, V(\Omega)) \cap H^1(I, V(\Omega)^*) \times L^2(I, L^2_0(\Omega))$, the control $\boldsymbol{u} \in L^2(\Omega \times I)^2$ and the adjoint state $\boldsymbol{q} = (\boldsymbol{\lambda}, \mu) \in L^2(I, V(\Omega)) \cap$

$H^1(I, V(\Omega)^*) \times L^2(I, L^2_0(\Omega))$, given in the strong form, characterizes the minimizer.

$$
\begin{aligned}
\boldsymbol{v}_t - \triangle \boldsymbol{v} + \nabla p &= \boldsymbol{u} + \boldsymbol{z} && \text{in } Q\\
-\operatorname{div} \boldsymbol{v} &= 0 && \text{in } Q\\
\boldsymbol{v} &= \mathbf{0} && \text{on } \Gamma\\
\boldsymbol{v} &= \boldsymbol{v}_0 && \text{at } t = 0\\
-\boldsymbol{\lambda}_t - \triangle \boldsymbol{\lambda} + \nabla \mu &= \boldsymbol{v} - \boldsymbol{v}_d && \text{in } Q\\
-\operatorname{div} \boldsymbol{\lambda} &= 0 && \text{in } Q\\
\boldsymbol{\lambda} &= \mathbf{0} && \text{on } \Gamma\\
\boldsymbol{\lambda} &= \mathbf{0} && \text{at } t = T\\
\boldsymbol{u} &= -\frac{1}{\beta}\boldsymbol{\lambda} && \text{in } Q
\end{aligned}
$$

Supplementary Material

The optimal state, adjoint state, and control are known analytically, noting that the pressure p and the adjoint pressure μ are normalized by having mean-value zero:

$$
\begin{aligned}
\boldsymbol{v} &= \left[(\zeta + \eta\, e^t)\, \sin^2(\pi\, x_1)\, \sin(2\,\pi\, x_2),\; -(\zeta + \eta\, e^t)\, \sin(2\,\pi\, x_1)\, \sin^2(\pi\, x_2)\right]^\top\\
p &= -\pi\, (\zeta + \eta\, e^t)\, \sin(2\,\pi\, x_1)\, \sin(2\,\pi\, x_2)\\
\boldsymbol{\lambda} &= -\left[-(e^T - e^t)\, \sin^2(\pi\, x_1)\, \sin(2\,\pi\, x_2),\; (e^T - e^t)\, \sin(2\,\pi\, x_1)\, \sin^2(\pi\, x_2)\right]^\top\\
\mu &= -(x_1 + x_2)\\
\boldsymbol{u} &= \frac{1}{\beta}\left[-(e^T - e^t)\, \sin^2(\pi\, x_1)\, \sin(2\,\pi\, x_2),\; (e^T - e^t)\, \sin(2\,\pi\, x_1)\, \sin^2(\pi\, x_2)\right]^\top
\end{aligned}
$$

Notice that the sign of $(\boldsymbol{\lambda}, \mu)$ is reversed in [Güttel and Pearson, 2017, Section 6.2]. Consequently, the control law reads $\boldsymbol{u} = \frac{1}{\beta}\boldsymbol{\lambda}$ in [Güttel and Pearson, 2017, Section 6.2].

References

S. Güttel and J. W. Pearson. A rational deferred correction approach to parabolic optimal control problems. *IMA Journal of Numerical Analysis*, online-first, 2017. doi: 10.1093/imanum/drx046. URL https://academic.oup.com/imajna/advance-article/doi/10.1093/imanum/drx046/4372128.