Introduction

We have a simple example of a time-optimal control problem subject to the linear heat equation and pointwise bound constraints on the control. The goal is to steer the heat equation into an L^2-ball centered at some desired state in the shortest time possible by an appropriate choice of the control. The time-optimal control problem can be transformed to a fixed time interval and both versions are given below.

This particular problem utilizes a control function varying in time only. The exact solution is unknown, but numerical values are provided.

The problem has been used as numerical test in [Bonifacius et al., 2018a, Example 5.2].

Variables & Notation

Unknowns

$q \in Q = L^\infty((0,T);\mathbb{R}^2)$	control variable
$u \in U = H^1((0,T);L^2(\Omega)) \cap L^2((0,T);H_0^1(\Omega) \cap H^2(\Omega))$	state variable
T	terminal time

Given Data

$\Omega = (0,1)^2$	spatial domain
$u_d = 0 \in H_0^1(\Omega)$	desired state
$\delta_0 = \dfrac{1}{10} > 0$	tolerance to desired state
$\alpha \geq 0$	control cost parameter (arbitrary)
$u_0 = 4\sin(\pi x_1^2)\sin(\pi x_2^3)$	initial state
$c = 0.03$	coefficient in the PDE
$q_a = -1.5$	lower control bound
$q_b = 0$	upper control bound
$\omega_1 = (0,0.5) \times (0,1)$	control domain 1
$\omega_2 = (0.5,1) \times (0,0.5)$	control domain 2

The control-action operator is defined as

$$
\begin{aligned}
B\colon \mathbb{R}^2 &\to L^2(\Omega),\\
q = (q_1,q_2) &\mapsto Bq = q_1\chi_{\omega_1} + q_2\chi_{\omega_2}
\end{aligned}
$$

where χ_{ω_1} and χ_{ω_2} denote the characteristic functions on ω_1 and ω_2.

Problem Description

$$
\text{Minimize } j(T,q) := T + \frac{\alpha}{2}\int_0^T \|q(t)\|_{\mathbb{R}^2}^2\,\mathrm{d}t,
$$

$$
\text{subject to } \begin{cases} T > 0, & \\ \partial_t u - c\triangle u = Bq, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{in } \Omega, \\ \frac{1}{2}\|u(T)-u_d\|_{L^2(\Omega)}^2 - \frac{\delta_0^2}{2} \le 0, & \\ q_a \le q(t) \le q_b, & t\in(0,T). \end{cases} \tag{P}
$$

The state equation is transformed to the reference time interval $(0,1)$ in order to deal with the variable time horizon; see [Bonifacius et al., 2018a, Section 3.1] for details. Thus, the transformed version of (P) reads

$$
\text{Minimize } \widehat{j}(T,\widehat{q}) := T\left(1+\frac{\alpha}{2}\right)\int_0^1 \|\widehat{q}(t)\|_{\mathbb{R}^2}^2\,\mathrm{d}t,
$$

$$
\text{subject to } \begin{cases} T > 0, & \\ \partial_t \widehat{u} - Tc\triangle \widehat{u} = TBq, & \text{in } (0,1)\times\Omega, \\ \widehat{u} = 0, & \text{on } (0,1)\times\partial\Omega, \\ \widehat{u}(0) = u_0, & \text{in } \Omega, \\ \frac{1}{2}\|\widehat{u}(1)-u_d\|_{L^2(\Omega)}^2 - \frac{\delta_0^2}{2} \le 0, & \\ q_a \le \widehat{q}(t) \le q_b, & t\in(0,1). \end{cases} \tag{$\widehat{P}$}
$$

Note that the problems (P) and $(\widehat{P})$ are equivalent. The unknowns for the transformed problem $(\widehat{P})$ are $\widehat{q} \in \widehat{Q} = L^\infty((0,1);\mathbb{R}^2)$ and $\widehat{u} \in \widehat{U} = H^1((0,1);L^2(\Omega)) \cap L^2((0,1);H_0^1(\Omega)\cap H^2(\Omega))$.

Optimality System

The first-order necessary optimality conditions for $(\widehat{P})$ are formally given as follows: for given local minimizers $\overline{q} \in \widehat{Q}$ $\overline{u} \in \widehat{U}$, $\overline{T} > 0$ there exists Lagrange multipliers $\overline{\mu} > 0$ and $\overline{z} \in W(0,1) = \{v \in L^2(0,1;H_0^1(\Omega)) : \partial_t v \in L^2(0,1;H^{-1}(\Omega))\}$ such that

$$
\begin{aligned}
\int_0^1 1 + \frac{\alpha}{2}\|\overline{q}(t)\|_{\mathbb{R}^2}^2 + \big(B\overline{q}(t) + c\triangle\overline{u}(t),\, \overline{z}(t)\big)_{L^2(\Omega)}\,\mathrm{d}t = 0,& \\
\int_0^1 \overline{T}\big(\alpha\,\overline{q}(t) + B^*\overline{z}(t),\, q(t)-\overline{q}(t)\big)_{\mathbb{R}^2}\,\mathrm{d}t \ge 0, \quad \forall q_a \le q(t) \le q_b,& \\
\|\overline{u}(1) - u_d\|_{L^2(\Omega)} = \delta_0,&
\end{aligned}
$$

where the *adjoint state* $\overline{z} \in W(0,1)$ is determined by

$$-\partial_t \overline{z}(t) - \overline{T} \triangle \overline{z}(t) = 0, \quad t \in (0,1) \quad \overline{z}(1) = \overline{\mu}\left(\overline{u}(1) - u_d\right). \tag{0.1}$$

It can be shown that the above optimality conditions are satisfied in the given example, see, [Bonifacius et al., 2018a, Theorem 3.10].

Supplementary Material

For the example, no analytical solution is known. However, numerical values from [Bonifacius et al., 2018a, Example 5.2] are provided. The state and adjoint state equations are discretized by means of the discontinuous Galerkin scheme in time (corresponding to a version of the implicit Euler method) and linear finite elements in space. This scheme is guaranteed to converge with a rate $|\log k|(k + h^2)$ with k denoting the temporal mesh size and h the spatial mesh size; cf. [Bonifacius et al., 2018a, Corollary 4.16]. For further details on the implementation we refer to [Bonifacius et al., 2018a, Section 5].

The following table provides results for [Bonifacius et al., 2018a, Example 5.2] and they were provided by the authors for different values of the control cost parameter α, number of time steps M and number of spatial nodes N. The analysis for the case $\alpha = 0$ can be found in Bonifacius et al. [2018b].

		$\alpha = 10$	$\alpha = 1$	$\alpha = 0.1$	$\alpha = 0.01$	$\alpha = 0.001$	$\alpha = 0$
M	N	$\overline{T}$					
640	289	2.605661	2.075153	1.845201	1.808456	1.808257	1.808255
1280	1089	2.593450	2.061039	1.830766	1.794457	1.794261	1.794260
2560	4225	2.589968	2.057095	1.826762	1.790567	1.790372	1.790370
5120	16641	2.588884	2.055897	1.825559	1.789395	1.789200	1.789198
10240	16641	2.588670	2.055684	1.825355	1.789193	1.788998	1.788997

References

L. Bonifacius, K. Pieper, and B. Vexler. A priori error estimates for space-time finite element discretization of parabolic time-optimal control problems. *ArXiv e-prints*, February 2018a. URL https://arxiv.org/abs/1802.00611.

L. Bonifacius, K. Pieper, and B. Vexler. Error estimates for space-time discretization of parabolic time-optimal control problems with bang-bang controls. *ArXiv e-prints*, September 2018b. URL https://arxiv.org/abs/1809.04886.