Introduction

The problem at hand is an optimal control problem in which the state is determined by variational inequality, viz. the elliptic obstacle problem, rather than by a partial differential equation. In fact, the variational inequality is formulated equivalently as an elliptic equation plus a complementarity system. Consequently, the optimal control problem is a function space MPCC (mathematical program with equilibrium constraints).

The problem and its solution are taken from [Meyer and Thoma, 2013, Example 7.1].

Variables & Notation

Unknowns

$$\begin{aligned} u &\in L^2(\Omega) && \text{control variable} \\ y &\in H_0^1(\Omega) && \text{state variable} \\ \xi &\in L^2(\Omega) && \text{slack variable} \end{aligned}$$

Given Data

The given data is chosen in a way which admits an analytic solution.

$$\begin{aligned} &\Omega = (0,1)^2 && \text{computational domain} \\ &\Gamma && \text{its boundary} \\ &\Omega_1 \quad \text{(see below)} && \text{subdomain of } \Omega \\ &\Omega_2 = (0.0, 0.5) \times (0.0, 0.8) && \text{subdomain of } \Omega \\ &\Omega_3 = (0.5, 1.0) \times (0.0, 0.8) && \text{subdomain of } \Omega \end{aligned}$$

$$y_d(x) = \begin{cases} -400 \, \big(q_1(y_1) + q_2(y_2)\big)\big|_{y = Q^\top(x - \widehat{x}) + \widehat{x}}, & x \in \Omega_1 \\ z_1(x_1) \, z_2(x_2), & x \in \Omega_2 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{desired state (discontinuous)}$$

$$u_d(x) = \begin{cases} p_1(Q^\top(x - \widehat{x}) + \widehat{x}), & x \in \Omega_1 \\ -z_1''(x_1) - z_2''(x_2), & x \in \Omega_2 \\ -z_1(x_1 - 0.5) \, z_2(x_2), & x \in \Omega_3 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{desired control (discontinuous)}$$

The subdomain Ω_1 is a square with midpoint $\widehat{x} = (0.8, 0.9)$ and edge length 0.1, which has been rotated about its midpoint by 30 degrees in counter-clockwise direction. The four vertices of Ω_1 can thus be obtained from

$$\begin{pmatrix}\widehat{x} & \widehat{x} & \widehat{x} & \widehat{x}\end{pmatrix} + Q \begin{pmatrix} -0.05 & 0.05 & 0.05 & -0.05 \\ -0.05 & -0.05 & 0.05 & 0.05 \end{pmatrix} \approx \begin{pmatrix} 0.7817 & 0.8683 & 0.8183 & 0.7317 \\ 0.8317 & 0.8817 & 0.9683 & 0.9183 \end{pmatrix}$$

with the rotation matrix

$$Q = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}.$$

Note that Ω_1 does not intersect Ω_2 nor Ω_3. The remaining pieces of data are

$$\begin{aligned}
z_1(x_1) &= -4\,096\, x_1^6 + 6\,144\, x_1^5 - 3\,072\, x_1^4 + 512\, x_1^3 \\
z_2(x_2) &= -244.140\,625\, x_2^6 + 585.937\,500\, x_2^5 - 468.750\, x_2^4 + 125\, x_2^3 \\
q_1(y_1) &= -200\,(y_1 - 0.8)^2 + 0.5 \\
q_2(y_2) &= -200\,(y_2 - 0.9)^2 + 0.5 \\
p_1(y_1, y_2) &= q_1(y_1)\, q_2(y_2).
\end{aligned}$$

Problem Description

$$\begin{aligned}
&\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} \\
&\text{s.t.} \quad \begin{cases} -\triangle y = u + \xi & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \\ y \ge 0, \quad \xi \ge 0, \quad y\,\xi = 0 & \text{in } \Omega. \end{cases}
\end{aligned}$$

Optimality System

Besides the state $y \in H_0^1(\Omega)$, control $u \in L^2(\Omega)$ and slack variable $\xi \in L^2(\Omega)$, the optimality system consists of the adjoint state $p \in H_0^1(\Omega)$ and a Lagrange multiplier $\mu \in H^{-1}(\Omega)$ pertaining to the constraint $y \ge 0$. The adjoint state p serves a double role, since it also acts as Lagrange multiplier for the pointwise constraint $\xi \ge 0$. As usual for MPCCs, no multiplier is introduced for the constraint $y\,\xi = 0$.

It should be noted that for MPCCs, a canocical first-order optimality condition does not exist. The following system represents a particular set of first-order necessary conditions, viz. of strongly stationary type.

$$
\begin{aligned}
-\triangle y &= u + \xi && \text{in } \Omega \\
y &= 0 && \text{on } \partial\Omega \\
-\triangle p &= y - y_d + \mu && \text{in } \Omega \\
p &= 0 && \text{on } \partial\Omega \\
u - u_d - p &= 0 && \text{in } \Omega \\
y \ge 0, \quad \xi \ge 0, \quad y\,\xi &= 0 && \text{in } \Omega \\
\mu\, y &= 0 && \text{in } \Omega \text{ a weak sense} \\
p\,\xi &= 0 && \text{in } \Omega \\
p &\ge 0 && \text{in } B \\
\mu &\le 0 && \text{in } B \text{ in a weak sense.}
\end{aligned}
$$

The set $B = \{x \in \Omega : y(x) = \xi(x) = 0\}$ is termed the bi-active set. It is the last two conditions on the signs of p and μ which are particular for the concept of strong stationarity.

Since μ belongs only to $H^{-1}(\Omega)$, two of the conditions above must be imposed in a weak sense. This can be done in the following way:

$$
\begin{aligned}
\langle \mu, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= 0 \quad \text{for all } v \in H_0^1(\Omega) \text{ satisfying } v(x) = 0 \text{ where } y(x) = 0 \\
\langle \mu, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &\le 0 \quad \text{for all } v \in H_0^1(\Omega) \text{ satisfying } v(x) \ge 0 \text{ where } y(x) = 0 \\
&\qquad \text{and } v(x) = 0 \text{ where } \xi(x) > 0.
\end{aligned}
$$

Supplementary Material

The following functions given in [Meyer and Thoma, 2013, Example 7.1] satisfy the set of necessary optimality conditions of strongly stationary type above. An important feature of this selection is that there is a nontrivial bi-active set:

$$B = \{x \in \Omega : y(x) = \xi(x) = 0\} = (0.0, 1.0) \times (0.8, 1.0).$$

Moreover, second-order optimality conditions have been verified, and thus (y, ξ, u) is guaranteed to represent a local minimum.

$$y = \begin{cases} z_1(x_1)\, z_2(x_2), & x \in \Omega_2 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{(of class } C^2(\overline{\Omega})\text{)}$$

$$u = \begin{cases} -z_1''(x_1) - z_2''(x_2), & x \in \Omega_2 \\ -z_1(x_1 - 0.5)\, z_2(x_2), & x \in \Omega_3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\xi = \begin{cases} z_1(x_1 - 0.5)\, z_2(x_2), & x \in \Omega_3 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{(continuous)}$$

$$p = \begin{cases} p_1(Q^\top x), & x \in \Omega_1 \\ 0 & \text{elsewhere} \end{cases} \qquad \text{(continuous, but not } C^1(\Omega)\text{)}$$

$$\langle \mu, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \int_{\partial\Omega_1} \nabla p|_{\Omega_1} \cdot n_1\, v \,\mathrm{d}s,$$

where n_1 is the unit outer normal to the rotated square subdomain Ω_1. Note that μ is a line functional concentrated on $\partial\Omega_1$. In more explicit terms, it can be expressed as

$$\begin{aligned} \langle \mu, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = &\int_{0.75}^{0.85} Q \begin{pmatrix} -0.5\, q_1'(x_1) \\ 20\, q_1(x_1) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} v(x_1, 0.85) \,\mathrm{d}x_1 \\ &+ \int_{0.75}^{0.85} Q \begin{pmatrix} -0.5\, q_1'(x_1) \\ -20\, q_1(x_1) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} v(x_1, 0.95) \,\mathrm{d}x_1 \\ &+ \int_{0.85}^{0.95} Q \begin{pmatrix} 20\, q_2(x_2) \\ -0.5\, q_2'(x_2) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} v(0.75, x_2) \,\mathrm{d}x_2 \\ &+ \int_{0.85}^{0.95} Q \begin{pmatrix} -20\, q_2(x_2) \\ -0.5\, q_2'(x_2) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(0.85, x_2) \,\mathrm{d}x_2. \end{aligned}$$

The remaining data are

$$\begin{aligned} z_1''(x_1) &= -122\ 880\, x_1^4 + 122\ 880\, x_1^3 - 36\ 864\, x_1^2 + 3\ 072\, x_1 \\ z_2''(x_2) &= -7\ 324.218\ 750\, x_2^4 + 11\ 718.75\, x_2^3 - 5\ 625\, x_2^2 + 750\, x_2^1 \\ q_1'(x_1) &= -400\,(x_1 - 0.8) \\ q_2'(x_2) &= -400\,(x_2 - 0.9). \end{aligned}$$

Revision History

- 2021–02–11: fixed typo in transformation of data y_d and u_d on Ω_1
- 2013–03–01: problem added to the collection

References

C. Meyer and O. Thoma. A priori finite element error analysis for optimal control of the obstacle problem. *SIAM Journal on Numerical Analysis*, 51(1):605–628, 2013. doi: 10.1137/110836092.