Introduction

Here we present a simple distributed optimal control problem of the heat equation. Problems of this type are examined in detail within [Tröltzsch, 2010, Chapter 3]. The problem was derived as a test for the paper Güttel and Pearson [2017], which required optimal states and controls that are not polynomial in spatial or time variables. The problem is generically usable in dimensions 1 through 3 and maintains a parameter dependence for the regularization parameter β to serve as a test case for the β dependence of solvers.

The 2d version of this problem and analytical solution appear in [Güttel and Pearson, 2017, Section 6.1], where computations for $\beta = 0.05$ and final times $T = 1$ were conducted. The implementation is provided for $d = 2$ as well.

Variables & Notation

Unknowns

$u \in L^2(\Omega \times I)$	control variable
$y \in W(0,T) := L^2(I; H_0^1(\Omega)) \cap H^1(I; H^{-1}(\Omega))$	state variable

Given Data

$d \in \{1,2,3\}$	dimension of the problem
$T > 0$	length of time interval
$\Omega = (-1,1)^d$	spatial domain
$I = (0,T)$	time interval
$Q = (-1,1)^d \times (0,T)$	space-time domain
$\Sigma = \partial\Omega \times (0,T)$	lateral boundary of Q
$\beta > 0$	regularization parameter
$y_{d,T} = \left[\dfrac{d\,\pi^2}{4} + \dfrac{4}{d\,\pi^2\,\beta}\right] e^T + \left[1 - \dfrac{d\,\pi^2}{4} - \dfrac{4}{(4 + d\,\pi^2)\,\beta}\right] e^t$	aux. function for desired state
$y_d = y_{d,T}(t) \displaystyle\prod_{k=1}^d \cos\left(\frac{\pi\,x_k}{2}\right)$	desired state
$y_0 = \left(\dfrac{4}{d\,\pi^2\,\beta}\,e^T - \dfrac{4}{(4 + d\,\pi^2)\,\beta}\right) \displaystyle\prod_{k=1}^d \cos\left(\frac{\pi\,x_k}{2}\right)$	initial state

Problem Description

$$\text{Minimize} \quad \frac{1}{2}\int_I \int_\Omega (y - y_d)^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{\beta}{2}\int_I \int_\Omega u^2 \,\mathrm{d}x \,\mathrm{d}t$$

$$\text{s.t.} \quad \begin{cases} y_t - \triangle y = u & \text{in } Q \\ y = 0 & \text{on } \Sigma \\ y = y_0 & \text{at } t = 0 \end{cases}$$

Optimality System

The following optimality system for the control $u \in L^2(\Omega \times I)$, the state $y \in W(0,T)$, and the adjoint state $p \in W(0,T)$, given in the strong form, characterizes the unique minimizer.

$$\begin{aligned} y_t - \triangle y &= u && \text{in } Q \\ y &= 0 && \text{on } \Sigma \\ y &= y_0 && \text{at } t = 0 \\ -p_t - \triangle p &= y - y_d && \text{in } Q \\ p &= 0 && \text{on } \Sigma \\ p &= 0 && \text{at } t = T \\ u &= -\frac{1}{\beta}\, p && \text{in } Q \end{aligned}$$

Supplementary Material

The optimal state, adjoint state, and control are known analytically:

$$\begin{aligned} y &= \left(\frac{4}{d\,\pi^2\,\beta}\, e^T - \frac{4}{(4 + d\,\pi^2)\,\beta}\, e^t \right) \prod_{k=1}^{d} \cos\left(\frac{\pi\, x_k}{2}\right) \\ p &= -(e^T - e^t) \prod_{k=1}^{d} \cos\left(\frac{\pi\, x_k}{2}\right) \\ u &= \frac{1}{\beta}\,(e^T - e^t) \prod_{k=1}^{d} \cos\left(\frac{\pi\, x_k}{2}\right) \end{aligned}$$

Notice that the sign of p is reversed in [Güttel and Pearson, 2017, Section 6.1]. Consequently, the control law reads $u = \frac{1}{\beta}\, p$ in [Güttel and Pearson, 2017, Section 6.1].

References

S. Güttel and J. W. Pearson. A rational deferred correction approach to parabolic optimal control problems. *IMA Journal of Numerical Analysis*, online-first, 2017. doi: 10.1093/imanum/drx046. URL https://academic.oup.com/imajna/advance-article/doi/10.1093/imanum/drx046/4372128.

F. Tröltzsch. *Optimal Control of Partial Differential Equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, 2010.